STABILITY OF RODS FROM INHOMOGENEOUSLY AGING MATERIAL UNDER NONLINEAR CREEP CONDITIONS

A. D. Drozdov

UDC 539.376

Stability conditions are obtained for a bonded rod fabricated from an inhomogeneously aging material with a nonlinear creep law.

The stability problem of inhomogeneously aging viscoelastic rods was investigated in a linear formulation in [1, 2].

## 1. FORMULATION OF THE PROBLEM

Let us consider the bending of a rectilinear rod of length $\ell$ fabricated from an inhomogeneously aging viscoelastic material. The rod has two axes of symmetry. The bending occurs in a plane passing through the longitudinal axis and an axis of symmetry. Let us introduce an Ox axis directed along the rod longitudinal axis in the undeformed state. The rod transverse section is identical for all points x. We introduce axes $x_1$ and $x_2$ in the rod section. The $x_1$ axis lies in the plane of rod bending, while the $x_2$ axis is directed along the neutral axis. We denote the domain in the $x_1x_2$ plane that is occupied by the rod section by $\Omega$. The area S of the rod transverse section, and the moment of inertia of the section with respect to the neutral axis J are

$$\int_\Omega ds = S, \quad \int_\Omega x_1 ds = 0, \quad \int_\Omega x_1^2 ds = J. \tag{1.1}$$

Here ds is a section area element.

We set the beginning of time measurement at the time of material formation in the neighborhood of the point O. We denote the age of the material in the neighborhood of the point x with respect to the material at the point O by $\rho(x)$. The function $\rho$ is piecewise-continuous and bounded.

At the time $t_0 \geqq 0$, a compressive force P and a distributed transverse load of intensity q(x) are applied to the rod. For a uniaxial stress state, the stress $\sigma(t, x)$ and the strain $e(t, x)$ at the point x at time $t \geq t_0$ are connected by the relationship [3]

$$E\varphi(e(t, x)) = (I + K)\sigma, \quad \sigma(t, x) = E(I - R)\varphi(e), \tag{1.2}$$

where E is the constant instantaneous elastic strain modulus, I is a unit operator, K and R are the creep and relaxation operators

$$K\sigma = \int_{t_0}^{t} k(t + \rho(x), \tau + \rho(x))\, \sigma(\tau, x)\, d\tau, \quad Re = \int_{t_0}^{t} r(t + \rho(x), \tau + \rho(x))\, e(\tau, x)\, d\tau;$$

$k(t, \tau)$ and $r(t, \tau)$ are the creep and relaxation kernels, and $\varphi$ is a given piecewise-continuous bounded function. These quantities are determined from simple creep and relaxation tests.

Let us note that recent experimental investigations [4, 5] indicate that (1.2) can be applied uniformly for monotonic or nonmonotonic changes in the strain in time for certain polymers. It has also been established that the equation of state (1.2) describes the results of experiments well for step and contrast loading of polyvinyl chloride and polymethyl methacrylate specimens.

Furthermore, let there be a function $r_1(t, \tau)$ such that

$$|r_1| = \sup_t \int_{t_0}^{t} r_1(t, \tau)\, d\tau < 1, \quad t \geqslant t_0$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 150-155, July-August, 1985. Original article submitted May 24, 1984.



0021-8944/85/2604-0592$09.50 

and for all $0 \leqq x \leqq \ell$, $t_0 \leqq \tau \leqq t$

$$0 \leqslant r(t+\rho(x),\ \tau+\rho(x)) \leqslant r_1(t,\ \tau).$$

The function $r_1(t, \tau)$ allows of the representation

$$r_1(t,\ \tau) = \psi_1(t,\ \tau) + \psi_2(t,\ \tau)(t-\tau)^{-\varkappa},$$

where the functions $\psi_1$, $\psi_2$ are continuous in t, $\tau$ and $0 < \kappa < 1$; there also exists a function $r_0(t, \tau)$ such that $|r_0| < 1$ and uniformly in $t \geqq t_1$ as $t_1 \to \infty$,

$$\lim_{t_1} \int_{t_1}^{t} \sup_x |r(t+\rho(x),\ \tau+\rho(x)) - r_0(t,\ \tau)|\, d\tau = 0.$$

Let $R_0$ denote the relaxation operator with kernel $r_0$, $K_0$ its corresponding creep operator, and $k_0$ the kernel of this operator. It is considered that $|k_0| < \infty$.

## 2. EQUATIONS FOR THE ROD DEFLECTION

Let $w_1(t, x, x_1)$ and $w_2(t, x, x_1)$ be the longitudinal displacement and deflection at a point of the rod that is at a distance $x_1$ from the longitudinal axis. In conformity with the hypothesis of plane sections

$$w_1 = u(t,\ x) - x_1 y'(t,\ x),\ w_2 = y(t,\ x),\ y' = \partial y/\partial x, \tag{2.1}$$

where u and y are the longitudinal displacement and deflection at a point on the rod longitudinal axis.

It follows from the relationships (2.1) that for small strains

$$e = u' - x_1 y''. \tag{2.2}$$

Let M(t, x) denote the bending moment, and N(t, x) the normal force, while Q(t, x) is the transverse force

$$M = -\int_\Omega \sigma x_1 ds,\ N = -\int_\Omega \sigma ds. \tag{2.3}$$

Substituting (1.2) and (2.2) into (2.3), we obtain

$$M = -E(I-R)\int_\Omega \varphi(u' - x_1 y'') x_1 ds,\ N = -E(I-R)\int_\Omega \varphi(u' - x_1 y'')\, ds. \tag{2.4}$$

Considering the loading process to be sufficiently slow, we neglect inertial forces. We assume, moreover, that the rod deflection is sufficiently small, so that the quantity $(y')^2$ can be neglected in comparison with one. Then the equilibrium equations for a rod element have the form [6]

$$N' = 0,\ M' = Q,\ Q' = -Ny'' + q. \tag{2.5}$$

Let $u_0$, $y_0$ denote the displacement of points of the rod axis and $M_0$, $N_0$, $Q_0$ the bending moment and longitudinal and transverse forces in the absence of a transverse load ($q = 0$).

We set

$$y_0 = 0,\ M_0 = 0,\ N_0 = P,\ Q_0 = 0. \tag{2.6}$$

The longitudinal displacement $u_0$ is determined from the relationships (2.4), (2.6) with (1.1) taken into account:

$$\varphi(u_0') = -(I+K)\,P/(ES). \tag{2.7}$$

Let the transverse load intensity q be sufficiently small. We set

$$u = u_0 + \Delta u,\ y = y_0 + \Delta y,\ M = M_0 + \Delta M,\ N = N_0 + \Delta N,\ Q = Q_0 + \Delta Q \tag{2.8}$$

and we consider the increments in the displacements, forces, and moments that are caused by the transverse load to be sufficiently small also. We substitute (2.8) into the relationships (2.4) and (2.5). Taking account of (1.4) and (2.6), and neglecting products of quantities with the indicator Δ, we have

$$(\Delta N)' = 0,\ (\Delta M)' = \Delta Q,\ (\Delta Q)' = -P(\Delta y)'' + q; \tag{2.9}$$

$$\Delta M = EJ\,(I - R)\,\varphi'\left(u_0'\right)(\Delta y)'',\ \Delta N = -ES\,(I - R)\,\varphi'\left(u_0'\right)(\Delta u)'. \tag{2.10}$$

Definition. A rod is called Lyapunov stable in an infinite time interval if for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that from the inequality $\sup_x|q(x)| < \delta$ there follows the estimate $\sup_{t,x}|\Delta y(t, x)| < \varepsilon$ $(0 \leqq x \leqq \ell,\ t \geqq t_0)$.

## 3. DERIVATION OF THE STABILITY CONDITIONS

Since the derivation of the stability conditions is analogous for distinct kinds of rod end fixings, we limit ourselves to the case of a rod whose ends are rigidly clamped: $y(t, 0) = y(t, \ell) = y'(t, 0) = y'(t, \ell) = 0$. From this relationship and (2.6), (2.8) we obtain

$$\Delta y(t,\ 0) = \Delta y(t,\ l) = \Delta y'(t,\ 0) = \Delta y'(t,\ l) = 0. \tag{3.1}$$

According to (2.9) and (2.10), the rod equilibrium equation can be written in the form

$$EJ\left[(I - R)\,\varphi'\left(u_0'\right)(\Delta y)''\right]'' + P\,(\Delta y)'' = q. \tag{3.2}$$

Let us multiply this equation by $\Delta y(t, x)$ and integrate with respect to x between the limits 0 and $\ell$. Integrating by parts and taking account of the boundary conditions (3.1), we find

$$\int_0^l (\Delta y)''\,\varphi'\left(u_0'\right)(\Delta y)''\,dx = \int_0^l (\Delta y)''\,R\varphi'\left(u_0'\right)(\Delta y)''\,dx + \alpha\int_0^l ((\Delta y)')^2\,dx + \int_0^l q_1\Delta y\,dx,\ \alpha = P/(EJ),\ q_1 = q/(EJ). \tag{3.3}$$

Let us estimate the terms in (3.3) by considering that for any $x \in [0, l]$

$$0 < c_1 \leqslant \varphi'\left(u_0'\right) \leqslant c_2 < \infty.$$

Using the Cauchy–Bunyakovskii inequality, we have

$$\begin{gathered} c_1 Y_2^2(t) \leqslant Y^2(t) = \int_0^l \varphi'\left(u_0'\right)[(\Delta y)'']^2\,dx \leqslant c_2 Y_2^2(t), \\ \left|\int_0^l (\Delta y)''\,R\varphi'\left(u_0'\right)(\Delta y)''\,dx\right| \leqslant c_2 Y_2(t)\int_{t_0}^t r_1(t,\ \tau)\,Y_2(\tau)\,d\tau,\ \left|\int_0^l q_1\Delta y\,dx\right| \leqslant G Y_0(t) \end{gathered} \tag{3.4}$$

where

$$G^2 = \int_0^l q_1^2\,dx;\ Y_j^2 = \int_0^l\left[\frac{\partial^j}{\partial x^j}\Delta y\,(t,\ x)\right]^2 dx \quad (j = 0,\ 1,\ 2).$$

Let U denote a set of functions v(x) that have square-integrable second derivative and satisfy the boundary conditions $v(0) = v'(0) = v(\ell) = v'(\ell) = 0$. We set

$$\lambda_0(t) = \inf_v \int_0^l \varphi'\left(u_0'\right)(v'')^2\,dx\left[\int_0^l v^2\,dx\right]^{-1},$$

$$\lambda_1(t) = \inf_v \int_0^l \varphi'\left(u_0'\right)(v'')^2\,dx\left[\int_0^l (v')^2\,dx\right]^{-1}.$$

Evidently $\lambda_0(t) \geqq \lambda_0^0 > 0$, $\lambda_1(t) \geqq \lambda_1^0 > 0$.

Let us introduce the notation

$$\Lambda_0^{-1} = \sup_t \lambda_0^{-1}(t),\ \Lambda_1^{-1} = \sup_t \lambda_1^{-1}(t),\quad t \geqslant t_0.$$

To estimate the quantities $Y_0(t)$, $Y_1(t)$ we use the inequalities

$$Y_0(t) \leqslant \Lambda_0^{-1/2} Y(t) \leqslant (c_2\Lambda_0^{-1})^{1/2} Y_2(t),\ Y_1^2(t) \leqslant \Lambda_1^{-1} Y^2(t). \tag{3.5}$$

Taking (3.4) and (3.5) into account, we obtain from (3.3)

$$(1 - \alpha\Lambda_1^{-1}) Y^2(t) \leqslant c_2 Y_2(t) \int_{t_0}^{t} r_1(t, \tau) Y_2(\tau)\, d\tau + G(c_2\Lambda_0^{-1})^{1/2} Y_2(t).$$

For $\alpha < \Lambda_1$ we find from this relationship and (3.4)

$$c_1(1 - \alpha\Lambda_1^{-1}) Y_2(t) \leqslant c_2 \int_{t_0}^{t} r_1(t, \tau) Y_2(\tau)\, d\tau + G(c_2\Lambda_0^{-1})^{1/2}.$$

According to the Gronwall–Bellman inequality, the estimate

$$Y_2(t) \leqslant Gf(t) \tag{3.6}$$

follows from this relationship, where f is a monotonically increasing, continuous function.

We rewrite (3.2) in the form

$$EJ\left[(I - R_0)\varphi'(u_0')(\Delta y)''\right]'' + P(\Delta y)'' = EJ\left[(R - R_0)\varphi'(u_0')(\Delta y)''\right]'' + q.$$

Since the operator $I - R_0$ is independent of the coordinate x, it can be taken out from under the derivative sign. Applying the operator $I + K_0$ to the relationship obtained, we have

$$\left[\varphi'(u_0')(\Delta y)''\right]'' + \alpha(I + K_0)(\Delta y)'' = (I + K_0)\left[(R - R_0)\varphi'(u_0')(\Delta y)''\right]'' + (I + K_0) q_1. \tag{3.7}$$

We multiply (3.7) by $\Delta y(t, x)$ and integrate with respect to x between the limits 0 and $\ell$. Integrating by parts and taking account of boundary conditions (3.1), we write

$$\begin{aligned}\int_0^l \varphi'(u_0')[(\Delta y)'']^2\, dx &= \alpha \int_0^l (\Delta y)'(I + K_0)(\Delta y)'\, dx + \int_0^l \Delta y (I + K_0) q_1 dx + \\ &+ \int_0^l (\Delta y)''(I + K_0)(R - R_0)\varphi'(u_0')(\Delta y)''\, dx.\end{aligned} \tag{3.8}$$

We estimate the first two terms in the right side of this relationship by using the Cauchy–Bunyakovskii inequality and (3.5)

$$\begin{aligned}&\left|\int_0^l (\Delta y)'(I + K_0)(\Delta y)'\, dx\right| \leqslant (1 + |k_0|)\Lambda_1^{-1} Z^2(t),\\ &\left|\int_0^l \Delta y (I + K_0) q_1 dx\right| \leqslant G(1 + |k_0|)(c_2\Lambda_0^{-1})^{1/2} Z_2(t).\end{aligned} \tag{3.9}$$

Here

$$Z_j(t) = \sup_\tau Y_j(\tau);\ Z(t) = \sup_\tau Y(\tau);\ t_0 \leqslant \tau \leqslant t.$$

It follows from the properties of the limit relaxation operator that for any $\varepsilon_1 > 0$ there is a $T(\varepsilon_1) > t_0$ such that for $t \geq T(\varepsilon_1)$

$$\int_{T(\varepsilon_1)}^{t} \sup_x |r(t + \rho(x),\ \tau + \rho(x)) - r_0(t, \tau)|\, d\tau < \varepsilon_1.$$

We estimate the third term in the right side of (3.8) for $t \geqq T(\varepsilon_1)$ by using this relationship and the Cauchy–Bunyakovskii inequality

$$\left|\int_0^l (\Delta y)'' (I + K_0)(R - R_0)\varphi'(u_0')(\Delta y)''\,dx\right| \leqslant c_2 Z_2(t)\left[(1+|k_0|)\times\right.$$
$$\left.\times(|r_0|+|r_1|)Z_2(T(\varepsilon_1)) + \varepsilon_1 Z_2(t)(1+|k_0|)\right]. \quad (3.10)$$

We obtain from the relations (3.8)-(3.10)

$$\left[1-\alpha(1+|k_0|)\Lambda_1^{-1}\right] Z^2(t) \leqslant c_2(1+|k_0|)Z_2(t)\left[(|r_0|+|r_1|)Z_2(T(\varepsilon_1)) + \varepsilon_1 Z_2(t)\right] + G(1+|k_0|)\left(c_2\Lambda_0^{-1}\right)^{1/2} Z_2(t).$$

If

$$\alpha < \Lambda_1(1+|k_0|)^{-1}, \quad (3.11)$$

the estimate

$$\left\{c_1\left(1-\alpha\Lambda_1^{-1}(1+|k_0|)\right) - c_2(1+|k_0|)\varepsilon_1\right\} Z_2(t) \leqslant (1+|k_0|)\left[c_2(|r_0|+|r_1|)Z_2(T(\varepsilon_1)) + G\left(c_2\Lambda_0^{-1}\right)^{1/2}\right] \quad (3.12)$$

then follows from this inequality and (3.4).

According to (3.11), there is a $\varepsilon_1 > 0$ such that the expression in the braces is positive. We select $T(\varepsilon_1)$ for the found $\varepsilon_1$. Then from the inequalities (3.6) and (3.12) there results that for any $\varepsilon > 0$ and for all $t \geqq t_0$,

$$Z_2(t) < \varepsilon \quad (3.13)$$

for sufficiently small G. According to the boundary conditions

$$|\Delta y(t,x)| = \left|\int_0^x (x-\xi)(\Delta y(t,\xi))''\,d\xi\right| \leqslant (l/3)^{3/2} Z_2(t), \quad (3.14)$$

there follows from inequalities (3.13) and (3.14) the following theorem.

THEOREM. Let $\alpha < \Lambda_1(1 + |k_0|)^{-1}$. Then the rod is stable in an infinite time interval.

## 4. CERTAIN REMARKS AND PARTICULAR CASES

1) Stability conditions can also be obtained analogously for other types of rod end fixings: hinge-supported rod ends; one end rigidly clamped, the other end hinge-supported; one end rigidly clamped, the other end free.

The rod stability condition has the form $P < \Lambda_1 EJ(1 + |k_0|)^{-1}$. The quantity $\Lambda$ is determined from the solution of the variational problem in a set of functions U satisfying the appropriate boundary conditions.

2) If the function $\varphi$ is linear $[\varphi(\varepsilon) = \varepsilon]$, then the stability conditions obtained agree with the conditions presented in [2].

3) For the creep kernel [7]

$$k(t,\tau) = -E\frac{\partial}{\partial\tau}\left[\varphi_0(\tau)\left(1-e^{-\gamma(t-\tau)}\right)\right]$$

the limit creep kernel has the form

$$k_0(t,\tau) = -E\frac{\partial}{\partial\tau}\left[C_0\left(1-e^{-\gamma(t-\tau)}\right)\right],\; C_0 = \lim_\tau \varphi_0(\tau),\; \tau\to\infty.$$

In this case $|k_0| = EC_0$, and the rod stability condition takes the form $P < \Lambda_1 EJ(1 + EC_0)^{-1}$.

4) Let the rod be homogeneous ($\rho = 0$) and $\varphi(\varepsilon) = |\varepsilon|^\mu \operatorname{sign}\varepsilon$, $0 < \mu < 1$. We set

$$\lambda = \inf_v \int_0^l (v'')^2\,dx\left[\int_0^l (v')^2\,dx\right]^{-1},\quad v \in U.$$

The rod stability condition is

$$P < E\left[\lambda\mu J\,(1+|k_0|)^{-1}(1+|k|)^{\frac{\mu-1}{\mu}}S^{\frac{1-\mu}{\mu}}\right]^{\mu}.$$

We hence obtain for an elastic rod ($|k| = |k_0| = 0$)

$$P < E(\lambda\mu J)^{\mu}S^{1-\mu}.$$

This condition agrees with the stability condition for an elastic rod, computed by the method of a tangentially modular load [8].

## 5. STABILITY OF A BONDED NOLINEARLY VISCOELASTIC ROD SUBJECTED TO AGING

Let the rod be fabricated from a nonlinear viscoelastic material and be bonded by an elastic material. The armature is symmetric relative to the $x_1$ and $x_2$ axes.

The area of the armature transverse section is $S_a$ and the moment of inertia of the section of the bonding material is $J_a$. For the uniaxial stress state the relation between the stress and strain in the armature is described by Hooke's law $\sigma_a = E_a e_a$. Let the fundamental rod material be homogeneous ($\rho = 0$). We set

$$\beta = E_a J_a/(EJ),\ \Phi(u_0') = \varphi'(u_0') + \beta,$$
$$r^0(t,\tau) = \varphi'(u_0')\left[\Phi(u_0')\right]^{-1} r(t,\tau),\ \Lambda^{-1} = \sup_t [\lambda^0(t)]^{-1},$$
$$\lambda^0(t) = \inf_v \int_0^l \Phi(u_0')(v'')^2\,dx\left[\int_0^l (v')^2\,dx\right]^{-1}.$$

Let $k^0$ denote the limit creep kernel corresponding to the relaxational kernel $r^0$. The stability condition for the bonded rod has the form $P < EJ\Lambda(1 + |k^0|)^{-1}$.

The author is deeply grateful to N. Kh. Arutyunyan for attention to the research and for valuable remarks.

## LITERATURE CITED

1. N. Kh. Arutyunyan and V. B. Kolmanovskii, "On the stability of inhomogeneously aging viscoelastic rods," Prikl. Mat. Mekh., 43, No. 4 (1979).
2. N. Kh. Arutyunyan and V. B. Kolmanovskii, Creep Theory of Inhomogeneous Bodies [in Russian], Nauka, Moscow (1983).
3. Yu. N. Rabotnov, "Certain questions of creep theory," Vestn. Mosk. Univ., No. 10 (1948).
4. I. I. Bugakov and M. A. Chepovetskii, "Comparative investigation of nonlinear viscoelasticity equations," Izv. Akad. Nauk ArmSSR, Mekhanika, 37, No. 1 (1984).
5. A. Drescher and B. Mixhalski, "Rheological, mechanical, and optical properties of polymethyl methacryalte under conditions of a composite loading history," Mech. Teor. Stosow, 9, No. 2 (1971).
6. A. R. Rzhanitsyn, Equilibrium Stability of Elastic Systems [in Russian], Gostekhizdat, Moscow (1955).
7. N. Kh. Arutyunyan, Certain Questions of Creep Theory [in Russian], Gostekhizdat, Moscow (1952).
8. A. S. Vol'mir, Stability of Deformable Systems [in Russian], Nauka, Moscow (1967).